Complexity Theory: SV 3/3.
Advanced Topics

1 Instructions

- **Please submit your work at most 48 hours before your supervision to my cam email.**
- I'd appreciate it if you could typeset your work, but I'll accept legible scans of handwriting.
- Please make it clear which question you're writing a solution to, by referring to the numbering scheme of this sheet.
- I don't expect you to spend more than 3–4 hours of focussed work on each supervision's worth of work.
- The questions are not ordered by difficulty. If you're stuck on a question, feel free to ask me for a hint or wait till the supervision to discuss.
- Please never paste in answers that you don't understand – that defeats the purpose. It's not an issue if you leave an answer empty when you couldn't solve a problem.

2 Short questions – function classes

1. Give an example of an optimisation problem and a closely-related decision problem. Can you solve either of them efficiently given a black box for the other one?
2. Define FNP and FP using words.
3. Recall that $L \in \mathsf{NP}$ iff there's a polynomially-balanced, polynomial-time decidable relation R, such that $L = \{x \mid \exists y.\ R(x, y)\}$.
   (a) Define the terms 'polynomially-balanced' and 'polynomial-time decidable'.
   (b) Define the $\mathsf{CLIQUE}(G, k)$ decision problem in that form.
   (c) Give an example witness function for that language. Is your function polynomial-time computable?
4. If the $\mathsf{Factorisation}$ function was polynomial-time computable, would that imply $\mathsf{P} = \mathsf{NP}$?
5. Recall one of the four parts of the definition of a one-way function is a bound on the length $|f(x)|$ (see Lecture Notes).
   (a) Give an example of a function that would be one-way if that bullet point was removed from the definition.
   (b) Is your function one-way under the current definition?
6. How many accepting computations may an unambiguous NDTM have for an input x?
7. Justify briefly why $\mathsf{P} \subseteq \mathsf{UP} \subseteq \mathsf{NP}$.

3 Short questions – space complexity

1. Order the following classes into a sequence of the form '$\ldots \subseteq \ldots \subseteq \ldots$'.
   L, NL, NP, NPSPACE, P, PSPACE.

2. Prove that L is closed under complementation.
3. State the definition of a constructible function and give an informal argument for why both bullet points of the definition are desirable.
4. Give a summary of the proof that $\mathsf{NTIME}(f(n)) \subseteq \mathsf{SPACE}(\mathsf{f(n)})$.
5. Consider the nondeterministic algorithm used to prove that Reachability is in NL. Put a concrete bound on its space requirement, i.e. $\leq k \log n$ for some concrete k.
6. Summarise the proof that $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(k^{2f(n)})$.
7. State Savitch's Theorem and summarise its proof.
8. Make a table of the three algorithms for Reachability discussed in the lecture notes, for each of them noting: space complexity, time complexity, and what it was used to prove in the lecture notes. (Please justify the time complexity of the $\mathsf{SPACE}((\log n)^2)$ algorithm – this is the hard part of this question.) *PS I don't care if you present it as a table, or list of bullet points, or anything else that contains all the information.*

4 Short questions – hierarchy theorems & descriptive compl.

1. State the time hierarchy theorem and derive from it that $\mathsf{EXP} \neq \mathsf{P}$.
2. Consider the proof of the time hierarchy theorem from the lecture notes. The last sentence says 'We can now ask whether N accepts the input $[N]$, and we see that we get a contradiction either way.' Elaborate on this.
3. Expand the abbreviations and order the following logics in terms of expressiveness: FOL, SOL, ESO. What does Fagin's Theorem say?